Theory and Methodology

Linear programming model for finding optimal roadway grades that minimize earthwork cost

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Abstract

The cost of earthwork involved in road construction can vary widely based on the roadway grades chosen by the designer. There is almost an infinite number of feasible grades available to the designer to choose from, all of which satisfy the geometric specification of the road. For every feasible grade selected, a transportation problem must be solved, which is a very tedious task. The roadway grade selection is usually considered as a stage and the earthwork (transportation) is another stage. These two stages have always been treated separately in the literature. The model presented here combines the roadway grade selection stage and the earthwork transportation stage in a single linear programming problem, thus guaranteeing global optimality.

Keywords: Linear programming; Transportation; Earthwork allocation; Roadway grades

1. Introduction

The work involved in roadway design is usually handled in two stages. The first stage is related to the selection of roadway grades, while the second stage is that of minimizing the earthwork involved after stage one is completed. These two stages are usually completed separately from each other. A recent attempt to integrate them into one problem was done by Easa (1988). His approach was based on finding the complete enumeration of all technically feasible grades, calculating the cut and fill for each grade and then using linear programming to optimize the earthwork involved. This approach cannot be considered an integration of the two design stages in a true sense. It is actually a trial and error type of approach. Global optimality is guaranteed only through an exhaustive consideration of all possible feasible grades.

The approach presented in this paper combines the roadway grade selection stage with the earthwork allocation stage in a single linear programming problem. True integration of the two stages is achieved, and a direct solution that involves no trial and error is sought. The way the optimal solution is obtained is efficient, since only one linear program is considered, resulting in the optimum roadway grade and the optimum earth allocation (transportation).

2. The model

The designer must divide the road into a number of small sections. Following Easa's definition, a section is the portion of the road between two stations (Easa, 1988). Furthermore, the designer should decide on the number of grade lines in the road.
under study. In Fig. 1, the road under consideration is divided into 8 sections with two grade lines.

In order to represent the problem as a linear programming model, each section $i$ is considered as a point $(x_i, y_i)$ in the $x$-$y$ plane. The $y$-coordinate of this point is the average height of the ground at this section, and the $x$-coordinate is the center of this section.

3. Variable definitions

3.1. General parameters

$n = \text{number of sections in the road.}$
$m = \text{number of grade lines (linear segments of the final road).}$
$A_i = \text{surface area of section } i.$
$h_i = \text{the actual average height of section } i \text{ (as if it is leveled with zero slope).}$
$x_i = \text{the midpoint of section } i.$

3.2. Decision variables

$u_i = \text{the height of earth to be removed from section } i.$
$v_i = \text{the height of earth to be added to section } i.$
$T_{ij} = \text{number of cubic units of earth to be transported from section } i \text{ to section } j.$
$a_j = \text{the slope of grade line } j.$
$b_j = \text{the } y\text{-intercept of grade line } j.$

What is commonly used as parameters ($a_i$ and $b_i$) are now considered decision variables. Their values will be determined by the model that would result in the optimum leveling. Because they are unrestricted in sign they may be suited to linear programming with the following substitutions:

$$a_i = a_{i1} - a_{i2}, \quad b_i = b_{i1} - b_{i2},$$

where $a_{i1}, a_{i2}, b_{i1},$ and $b_{i2}$ are non-negative variables. This step may not be needed if the software used to solve the linear program has the feature to consider these variables as free variables taking either sign. In other word, the software will automatically do this step.

3.3. Cost parameters

$c_{ij} = \text{cost of transporting one cubic unit of earth from section } i \text{ to section } j.$
$p_i = \text{cost of excavating one cubic unit of earth from section } i.$
$q_i = \text{cost of embanking one cubic unit of earth to section } i.$

3.4. Grade line parameters

$L_{1j} = \text{the lower limit of the slope of grade line } j.$
$w_{1j} = \text{the upper limit of the slope of grade line } j.$
$L_{2j} = \text{the lower limit of the } y\text{-intercept value of grade line } j.$
$w_{2j} = \text{the upper limit of the } y\text{-intercept value of grade line } j.$
4. Problem description

The problem reduces to fitting each grade line (straight line) to some scattered points in the x–y plane using linear programming. This is very similar to using linear programming as a line fitting technique as done by Ignizio (1982). The components of such linear program is as follows:

The objective function that minimizes the total cost of embankment, excavation and haul is:

\[
\text{Minimize } Z = \sum_{i=1}^{n} (p_i A_i u_i + q_i A_i v_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} T_{ij}.
\]

The cost of excavation and that of embankment can both be added to the transportation cost, in which case the first term of the objective function can be dropped out.

The following set of constraints guarantees that whatever is transported to or from a section, is the result of excavation and embankment done to that section:

\[
\sum_{j=1}^{n} (T_{ij} - T_{ji}) = A_i (u_i - v_i) \quad i = 1, 2, \ldots, n.
\]

The next set of constraints says that the gap between the original height (elevation) of section i and the height of section i after final leveling of that section by grade line j is an exact result of the addition and removal that has been done to section i.

\[
h_i - (a_j x_i + b_j) = u_i - v_i \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m.
\]

The final road must be continuous, thus the end of every grade line must touch the beginning of the following grade line. Assuming grade line j2 meets grade line j1 at the point \(x_i\), then:

\[
a_{j1} x_i + b_{j1} = a_{j2} x_i + b_{j2}.
\]

The slope \(a_j\) for the grade line \(j\) may have some upper and lower limits, so does the variable \(b_j\) which represents the \(y\)-intercept of grade line \(j\).

\[
L_{1j} \leq a_j \leq w_{1j}, \quad L_{2j} \leq b_j \leq w_{2j}.
\]

After the model is put in standard linear programming form, the total number of columns (variables) is equal to \(n(n+1)+4m\), where the variables \(u_i\) and \(v_i\) account for \(2m\) columns and the variables \(a_j\) and \(b_j\) (after substitution, to accommodate the unrestricted in sign) account for \(4m\) columns. In addition to these, there are \(n(n-1)\) columns due to transportation variables.

The total number of rows is equal to \((2n+5m+1)\). Eqs. (2) and (3) account for \(n\) rows each, Eq. (5) accounts for \(4m\) rows and Eq. (4) accounts for \((m+1)\) rows, when including the two extreme ends of the road. Other equations may be required pertaining to the technical specifications of the road. One technical specification, say, the minimum height of the road at a certain section to avoid a soil susceptible to frost action.

5. Problem reduction

It is obvious that the transportation part of the model has increased the number of variables considerably. If the road does not involve a mountain at one end and a valley at the other (i.e. the expected grade-line levels are not radically different from the ground profile), the model lends itself to a greater reduction in the number of transportation variables. This can be achieved by restricting the movement of earth from any section to only a few surrounding sections. Suppose that the expected movement of earth will not exceed \(f\) sections away, then the number of transportation variables reduces (for \(n > f\)) from \(n(n-1)\) to \(2fn - f(f+1)\), with the number of variables eliminated equal to \((n-f)^2 - (n-f)\).

And the total number of variables for the complete problem is equal to \(2fn - f(f+1) + 4m\).

6. Examples

Two examples, (i) and (ii), are given here. Both are adopted from Easa (1988) in an attempt to replicate his results using the present model. There is some inconsistency in his examples between the heights of ground profile at different sections shown in the figures and the heights based on the amount of earth moved into or out of these sections. For exam-
example, the height at section 4 based on the figure is around 177 ft, but should be exactly 101.125 ft if calculated based on the amount of earth moved into section 4 (given that the length of the section is 500 ft and the width of the road is 24 ft, as reported). It should also be noted that the calculated heights in example (ii) are different from those heights calculated in example (i), although both have the same ground profile. The calculated heights are more accurate, and thus will be used here. The figures are left as they appeared in Easa, since they are not affecting the results.

To ensure that quantities of earth used here are the same as those in Easa, the problem was solved with the grade lines being fixed to the exact values of Easa. Namely, in example (i), the slopes are fixed to (-0.005) and (0.0025) for grade lines 1 and 2 respectively. It was found that quantities of earth are exactly the same as those in Easa. These quantities are used to find the values of the variables \( h \) in this paper.

6.1. Example (i). Two grade lines

The roadway profile is shown in Fig. 1. The costs as reported by Easa are:

Excavation Costs:
- common (between the sections) = $2.4/yd\(^3\)
- borrow (from borrow pit) = $2.7/yd\(^3\)

Embankment Cost:
- common (between the sections) = $5.4/yd\(^3\)
- borrow (from borrow pit) = $4.2/yd\(^3\)
- disposal (to landfill) = $0.6/yd\(^3\)

Haul:
- (across one section) = $1.05/yd\(^3\)

The horizontal coordinates \( x_i \) in feet (midpoints of the sections) are:

\[
\begin{align*}
x_1 &= 250, & x_2 &= 750, & x_3 &= 1250, \\
x_4 &= 1750, & x_5 &= 2250, & x_6 &= 2750, \\
x_7 &= 3250, & x_8 &= 3750.
\end{align*}
\]

The average heights \( h_i \) (in ft) at the various sections are not reported in Easa (1988), and are calculated here from his outputs:

\[
\begin{align*}
h_1 &= 234.062, & h_2 &= 217.973, & h_3 &= 118.564, \\
h_4 &= 101.125, & h_5 &= 180.938, & h_6 &= 306.343, \\
h_7 &= 297.85, & h_8 &= 198.007.
\end{align*}
\]

The capacity of each of the borrow pit and the landfill is 900,000 ft\(^3\).

Roadway Specifications:
- Width of road: 24 ft.
- Elevation of the beginning of the road = 210 ft.
- Elevation of the end of the road = 205 ft.
- Minimum height of fill at mid-road = 10 ft.

The problem can now be formulated as follows:

### 6.1.1. Objective Function

Min \( Z = \sum_{i=1}^{8} \sum_{j=1}^{8} c_{ij}T_{ij} + \sum_{i=1}^{8} c_{id}T_{id} + \sum_{i=1}^{8} c_{bi}T_{bi} \).

Note that \( c_{ij} \) is the collective cost of excavation from section \( i \), the cost of embankment in section \( j \) and the cost of transportation from section \( i \) to section \( j \), for example \( c_{13} = 2.4 + 5.4 + 2 \times 1.05 = $9.9/yd^3 \).

### 6.1.2. Constraints

\[
\begin{align*}
234.062 - \left[ 250(a_{11} - a_{12}) + (b_{11} - b_{12}) \right] &= u_1 - v_1, \\
217.973 - \left[ 750(a_{11} - a_{12}) + (b_{11} - b_{12}) \right] &= u_2 - v_2, \\
118.564 - \left[ 1250(a_{11} - a_{12}) + (b_{11} - b_{12}) \right] &= u_3 - v_3, \\
101.125 - \left[ 1750(a_{11} - a_{12}) + (b_{11} - b_{12}) \right] &= u_4 - v_4, \\
180.938 - \left[ 2250(a_{21} - a_{22}) + (b_{21} - b_{22}) \right] &= u_5 - v_5, \\
306.343 - \left[ 2750(a_{21} - a_{22}) + (b_{21} - b_{22}) \right] &= u_6 - v_6, \\
297.85 - \left[ 3250(a_{21} - a_{22}) + (b_{21} - b_{22}) \right] &= u_7 - v_7, \\
198.007 - \left[ 3750(a_{21} - a_{22}) + (b_{21} - b_{22}) \right] &= u_8 - v_8.
\end{align*}
\]

To force the two grade lines to touch at \( x = 2000 \):

\[
\begin{align*}
2000( a_{11} - a_{12} ) + ( b_{11} - b_{12} ) &= 2000( a_{21} - a_{22} ) + ( b_{21} - b_{22} ).
\end{align*}
\]
The elevation of the beginning of the road = 210 ft:

\[ 0(a_{11} - a_{12}) + (b_{11} - b_{12}) = 210. \]

The elevation of the end of the road = 205 ft:

\[ 4000(a_{21} - a_{22}) + (b_{21} - b_{22}) = 205. \]

The capacities of the borrow pit and landfill (ft³):

\[ T_{b1} + T_{b2} + T_{b3} + T_{b4} + T_{b5} + T_{b6} + T_{b7} + T_{b8} \leq 900,000, \]

\[ T_{l1} + T_{l2} + T_{l3} + T_{l4} + T_{l5} + T_{l6} + T_{l7} + T_{l8} \leq 900,000. \]

The quantity of soil excavated and embanked at a section is equal to the quantity transported from this section minus that transported into it (note, i ≠ j):

\[ A = 24\text{ft} \times 500\text{ft} = 12000 \text{ft}^2, \]

\[ 12000u_1 - 12000v_1 = \sum_{j=2}^{8} T_{1j} - \sum_{j=2}^{8} T_{1d} - T_{b1}, \]

\[ 12000u_2 - 12000v_2 = \sum_{j=1}^{8} T_{2j} - \sum_{j=1}^{8} T_{2d} - T_{b2}, \]

\[ 12000u_3 - 12000v_3 = \sum_{j=1}^{8} T_{3j} - \sum_{j=1}^{8} T_{3d} - T_{b3}, \]

\[ 12000u_4 - 12000v_4 = \sum_{j=1}^{8} T_{4j} - \sum_{j=1}^{8} T_{4d} - T_{b4}, \]

\[ 12000u_5 - 12000v_5 = \sum_{j=1}^{8} T_{5j} - \sum_{j=1}^{8} T_{5d} - T_{b5}, \]

\[ 12000u_6 - 12000v_6 = \sum_{j=1}^{8} T_{6j} - \sum_{j=1}^{8} T_{6d} - T_{b6}, \]

\[ 12000u_7 - 12000v_7 = \sum_{j=1}^{7} T_{7j} - \sum_{j=1}^{7} T_{7d} - T_{b7}, \]

\[ 12000u_8 - 12000v_8 = \sum_{j=1}^{7} T_{8j} - \sum_{j=1}^{7} T_{8d} - T_{b8}. \]

### 6.1.3. Results

The minimum value of the objective function is $1,044,032.79, with grade line parameters:

\[ a_1 = -0.00505806 \quad a_2 = 0.00255806 \]

\[ b_1 = 210 \quad b_2 = 194.76774 \]

The quantity of earth to be moved from one section to another:

\[ T_{1d} = 11,256.23 \text{yd}^3 \quad T_{73} = 30,592.30 \text{yd}^3 \]

\[ T_{23} = 5,229.58 \text{yd}^3 \quad T_{75} = 8,704.62 \text{yd}^3 \]

\[ T_{63} = 2,007.64 \text{yd}^3 \quad T_{78} = 2,822.44 \text{yd}^3 \]

\[ T_{64} = 44,454.84 \text{yd}^3 \]

The thickness of cuts and fills (u and v):

\[ u_1 = 25.3265 \text{ft} \quad v_3 = 85.1164 \text{ft} \]

\[ u_2 = 11.7666 \text{ft} \quad v_4 = 100.0234 \text{ft} \]

\[ u_6 = 104.5406 \text{ft} \quad v_5 = 19.5854 \text{ft} \]

\[ u_7 = 94.7686 \text{ft} \quad v_8 = 6.35048 \text{ft} \]

The model used in this example did not take into account the cost of operations done within a section. These costs, however, are present in sections 2, 5 and 8. To overcome this problem, one can find the cost incurred in these sections, and add it to the total cost. The cost for work done within the sections (i.e. sections 2, 5, and 8) is $42,657, when added to the total found in this linear program it becomes $1,086,693.99. The quantities reported in Easa are rounded off. His cost using these quantities is $1,086,694.50, which is very close to the cost found here.

The other method to include the cost of operations within the sections is to rerun the problem with each of these sections (2, 5, and 8) split into two sections, such that each section is entirely a cut or entirely a fill, thus, resulting in 11 sections.

When the problem was solved using 11 sections, a much better results are found as follow:

The minimum cost = $1,077,854.15.

Grade lines parameters are:

\[ a_1 = -0.00189269 \quad a_2 = -0.00060731 \]

\[ b_1 = 210 \quad b_2 = 207.42924 \]

The quantities to be moved (note that sections are numbered from 1 to 11) are:

\[ T_{13} = 3,779.25 \text{yd}^3 \quad T_{83} = 20,423.11 \text{yd}^3 \]

\[ T_{14} = 7,124.63 \text{yd}^3 \quad T_{93} = 26,489.00 \text{yd}^3 \]

\[ T_{15} = 7,953.61 \text{yd}^3 \quad T_{96} = 11,395.47 \text{yd}^3 \]

\[ T_{74} = 229.50 \text{yd}^3 \quad T_{911} = 3,175.66 \text{yd}^3 \]

\[ T_{84} = 24,276.36 \text{yd}^3 \quad T_{1011} = 251.23 \text{yd}^3 \]

It should be noted that the cost found using eleven sections is much less than the cost found by Easa.
6.2. Example (ii). Three grade lines

In this example, the roadway is divided into three grade lines as shown in Fig. 2, with the same data as in example (i). The results are:

The minimum value of the objective function = $608,682.7,

\[
\begin{align*}
    a_1 &= -0.018667 & b_1 &= 210 \\
    a_2 &= 0.031333 & b_2 &= 135 \\
    a_3 &= 0.024 & b_3 &= 301 \\
    T_{23} &= 10469.030 \text{ yd}^3 & T_{75} &= 7159.291 \text{ yd}^3 \\
    T_{24} &= 7839.656 \text{ yd}^3 & T_{78} &= 839.163 \text{ yd}^3 \\
    T_{64} &= 15838.130 \text{ yd}^3 & T_{1d} &= 17138.130 \text{ yd}^3 \\
    T_{65} &= 3409.876 \text{ yd}^3 & T_{2d} &= 3119.394 \text{ yd}^3
\end{align*}
\]

As noted above, the cost of operations done within the sections are not added. After adding the cost of operations done within the sections ($9115.5), the total cost becomes $617,798.2, which is slightly less than that reported by Easa ($618,068.00). The slopes of the three grade lines, however, are identical to those found by Easa.

This example was also solved, with the sections 2, 5 and 8 split into two sections each, resulting in 11 sections. The results are:

The value of the objective function = $618,566.9.

The grade line parameters are unchanged.

Quantities transported:

\[
\begin{align*}
    T_{23} &= 19.745 \text{ yd}^3 & T_{77} &= 100.149 \text{ yd}^3 \\
    T_{24} &= 10,469.027 \text{ yd}^3 & T_{910} &= 910.608 \text{ yd}^3 \\
    T_{25} &= 7,723.397 \text{ yd}^3 & T_{911} &= 8,359.164 \text{ yd}^3 \\
    T_{85} &= 15,954.383 \text{ yd}^3 & T_{1d} &= 17,138.138 \text{ yd}^3 \\
    T_{86} &= 3,440.017 \text{ yd}^3 & T_{2d} &= 3,247.7104 \text{ yd}^3 \\
    T_{96} &= 7,129.149 \text{ yd}^3
\end{align*}
\]

The value of the objective function is still less than that of Easa, but better than the 8 sections case. The slopes of the three grade lines remained identical to that of Easa, but gave an alternative solution to the quantities of earth moved.

7. Discussion

Easa’s method uses an iterative approach, where in each iteration, the slopes of the grade lines are incremented or decremented in value. At each iteration a linear transportation problem is solved. Therefore, it only gets closer to the optimum solution, depending on how small his iterative steps are, but can never guarantee global optimality.

This paper has presented a model that succeeded not only in reducing the number of times the problem is solved, but in arriving at global optimality in solving the problem only twice. The first run is to determine the cut and fill sections, the second run is
to find the optimum values of the grade lines and the optimum cost of transporting the cut and fill quantities. As can be seen in example (i) (with eleven sections), the solution found using the model presented here is much better than that found by Easa.

Perhaps the biggest advantage of this paper is avoiding the nonlinearity that is encountered by Easa's approach. A cut (or fill) in Easa's approach changes in height as well as in width for any change in the grade line, thus resulting in a nonlinear cost component in the objective function. This will lead to the possibility of having local optima, similar to what Easa had encountered in his paper. In this paper, however, the section is regarded as a point \((x_i, y_i)\) in the \(x-y\) plane representing a fixed width of a section and only the height is allowed to vary in the linear programming model. This implies that as the grade lines change, only the height of a section changes, resulting in a linear cost component in the objective function. The price to pay for this simplification is that a section may turn out to be both a cut and a fill at the same time, in which case the cost of work done inside the section is not included. However, it can be easily taken care of by one of two methods. The first method is by finding the cost incurred within this section separately, and this cost is then added to the total cost. The second method is splitting this section into two new sections, such that, one is entirely a cut and the other is entirely a fill, and running the problem again. Both methods are very effective as was demonstrated in the examples above.

References
