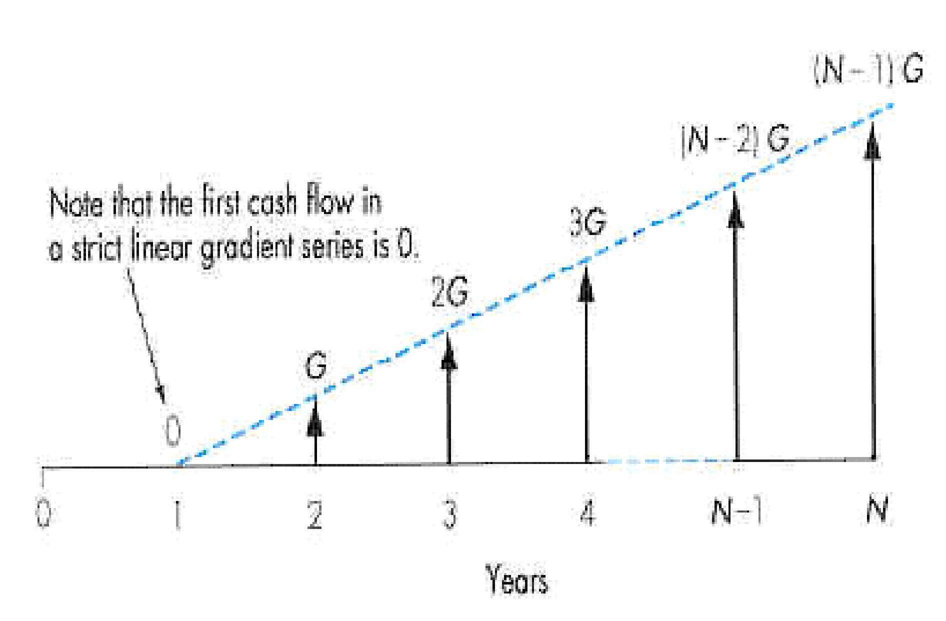
Gradient series

- Linear gradient
- Geometric gradient

Linear gradient series

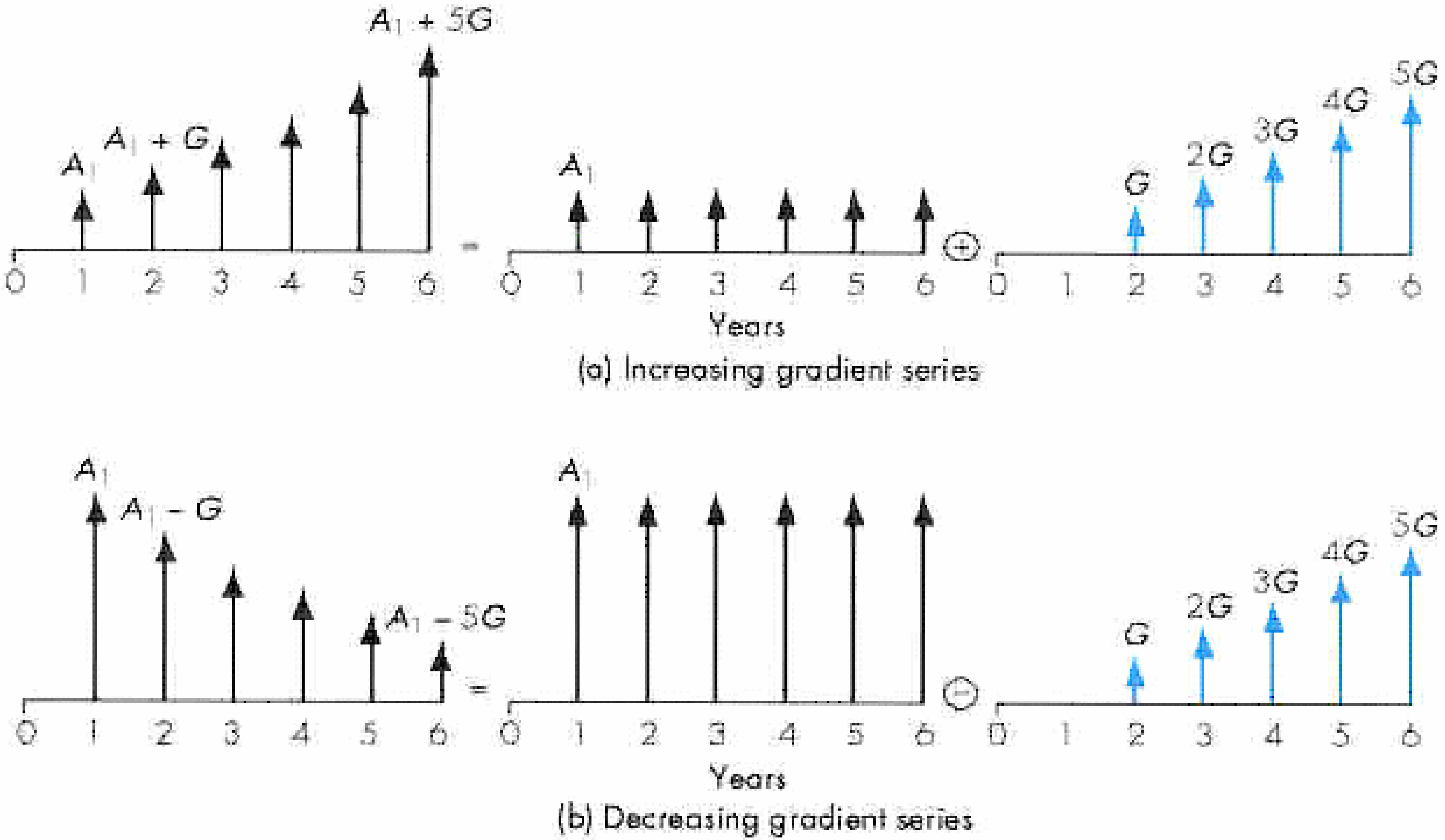


$$P = G \frac{(1 + i)^N - iN - 1}{i^2(1 + i)^N}$$

$$= G(P/G,N.i)$$

Gradient series present worth factor

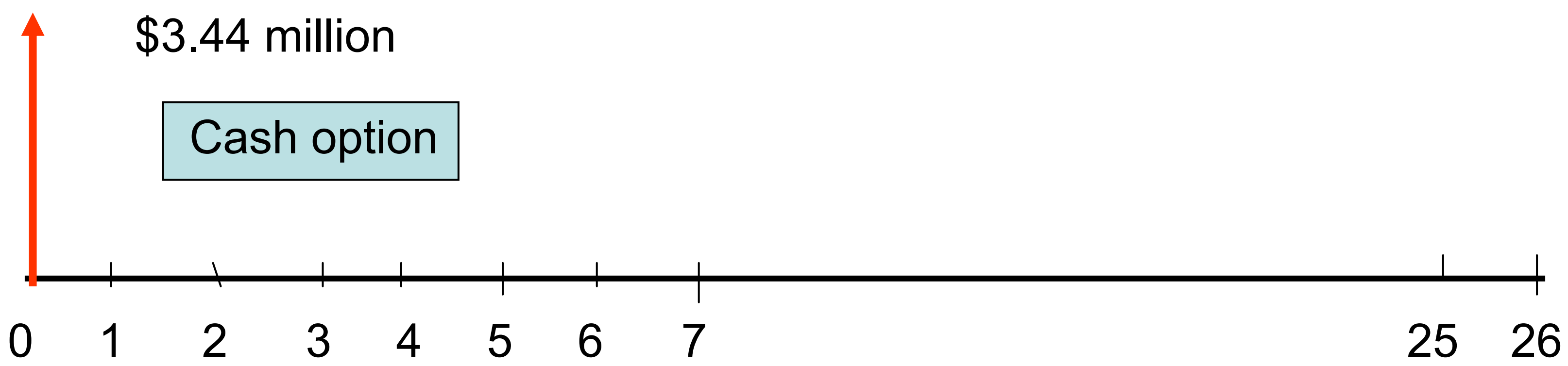
Gradient series as a composite series

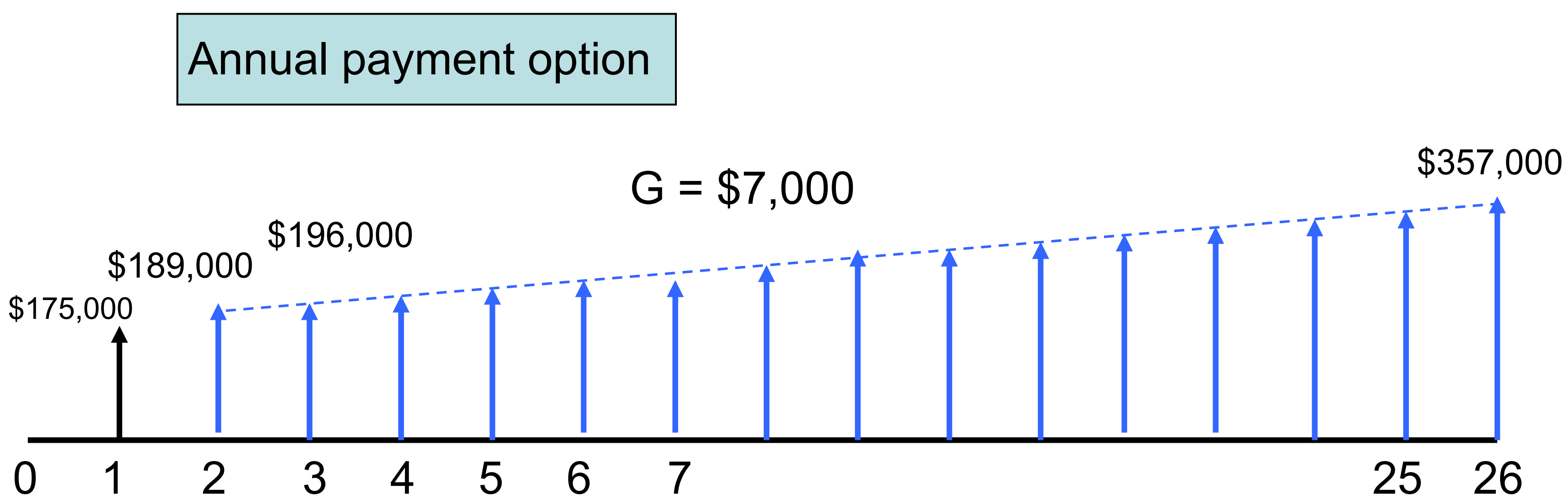


(a) Increasing gradient series

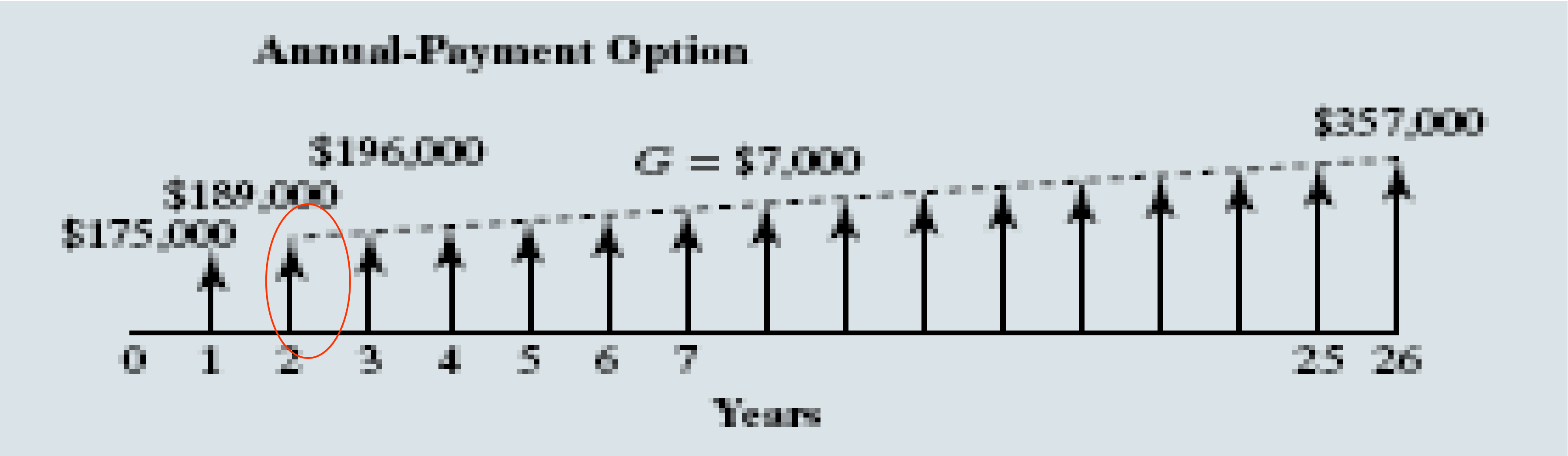
(b) Decreasing gradient series

Example – supper lottery





Equivalent present value of annual payment option at 4.5%

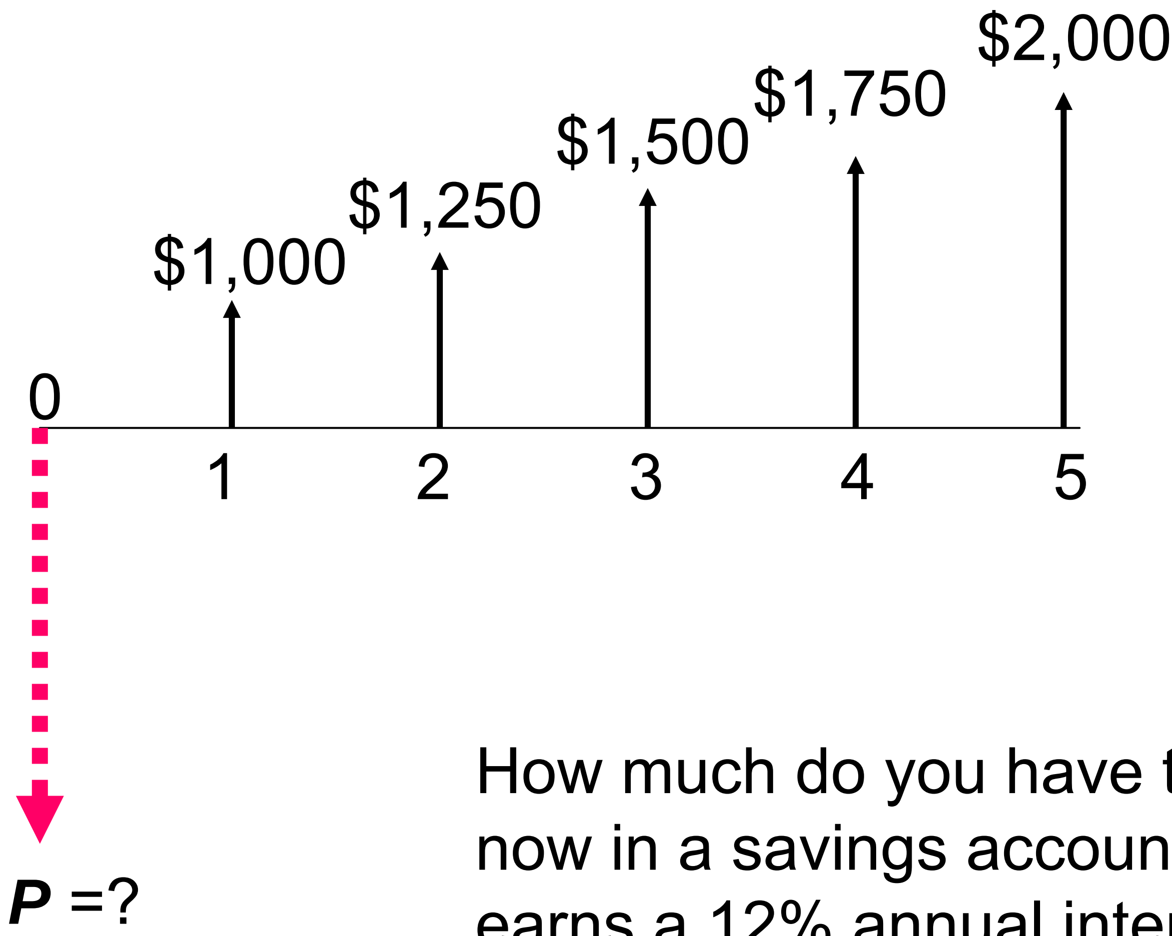


P = [$175,000 + $189,000(P/A, 4.5%, 25) + $7,000(P/G, 4.5%, 25)](P/F, 4.5%, 1) = $3,818,363

Excel solution

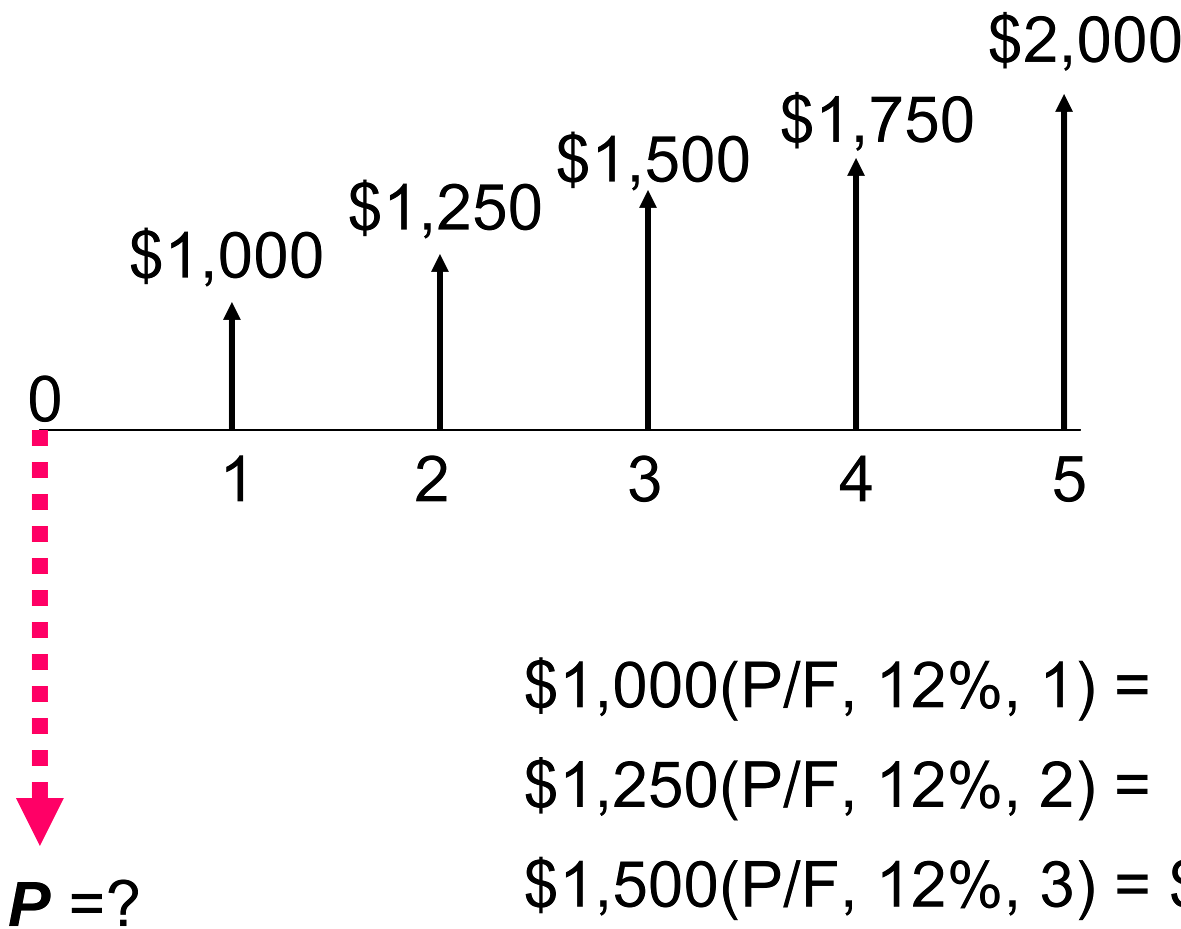
	A	B	C	D	E
1	Example 2.8 Cash Value Calculation Where the Payment Schedule				
2		Follows a Gradient Series			
3		Winning Jackpot	$ 7,000,000		
4		Interest rate (%)	4.5%		
5		Base amount	$ 189,000		
6		Gradient amount	$ 7,000		
7					
8	Payment	Payment Schedule	Annual Payment	Discounting	Present
9	Number	as % of Jackpot	before Taxes	Factor (4.5%)	Cash Value
10	1	2.5%	$ 175,000	0.9569	$ 167,464
11	2	2.7%	$ 189,000	0.9157	$ 173,073
12	3	2.8%	$ 196,000	0.8763	$ 171,754
13	4	2.9%	$ 203,000	0.8386	$ 170,228
14	5	3.0%	$ 210,000	0.8025	$ 168,515
15	6	3.1%	$ 217,000	0.7679	$ 166,633
16	7	3.2%	$ 224,000	0.7348	$ 164,602
17	8	3.3%	$ 231,000	0.7032	$ 162,436
18	9	3.4%	$ 238,000	0.6729	$ 160,151
19	10	3.5%	$ 245,000	0.6439	$ 157,762
20	11	3.6%	$ 252,000	0.6162	$ 155,282
21	12	3.7%	$ 259,000	0.5897	$ 152,723
22	13	3.8%	$ 266,000	0.5643	$ 150,096
23	14	3.9%	$ 273,000	0.5400	$ 147,413
24	15	4.0%	$ 280,000	0.5167	$ 144,682
25	16	4.1%	$ 287,000	0.4945	$ 141,913
26	17	4.2%	$ 294,000	0.4732	$ 139,114
27	18	4.3%	$ 301,000	0.4528	$ 136,293
28	19	4.4%	$ 308,000	0.4333	$ 133,457
29	20	4.5%	$ 315,000	0.4146	$ 130,613
30	21	4.6%	$ 322,000	0.3968	$ 127,766
31	22	4.7%	$ 329,000	0.3797	$ 124,922
32	23	4.8%	$ 336,000	0.3634	$ 122,086
33	24	4.9%	$ 343,000	0.3477	$ 119,262
34	25	5.0%	$ 350,000	0.3327	$ 116,456
35	26	5.1%	$ 357,000	0.3184	$ 113,670
37	Total	100.0%	$ 7,000,000		$ 3,818,363

Example – present value calculation for a gradient series



How much do you have to deposit now in a savings account that earns a 12% annual interest, if you want to withdraw the annual series as shown in the figure?

Method 1:



$1,000(P/F, 12%, 1) = $892.86

$1,250(P/F, 12%, 2) = $996.49

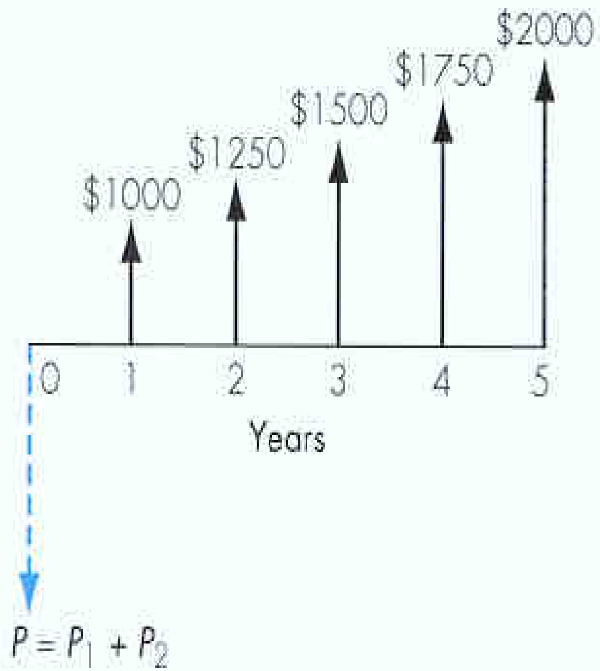
$1,500(P/F, 12%, 3) = $1,067.67

$1,750(P/F, 12%, 4) = $1,112.16

$2,000(P/F, 12%, 5) = $1,134.85

$5,204.03

Method 2:



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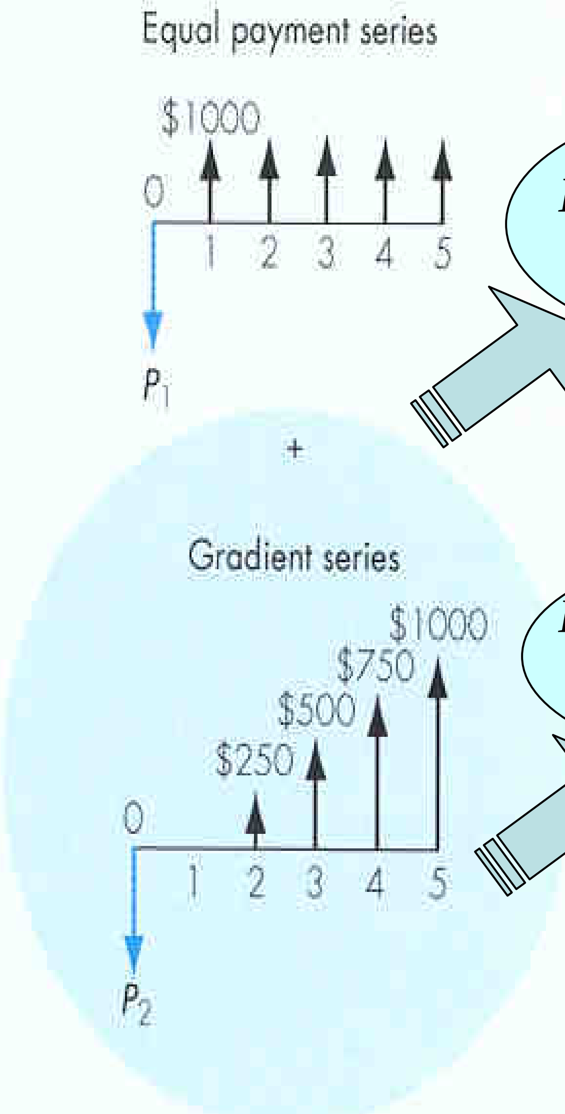


$$P_1 = \$1{,}000(P/A, 12\%, 5)$$
$$= \$3{,}604.80$$

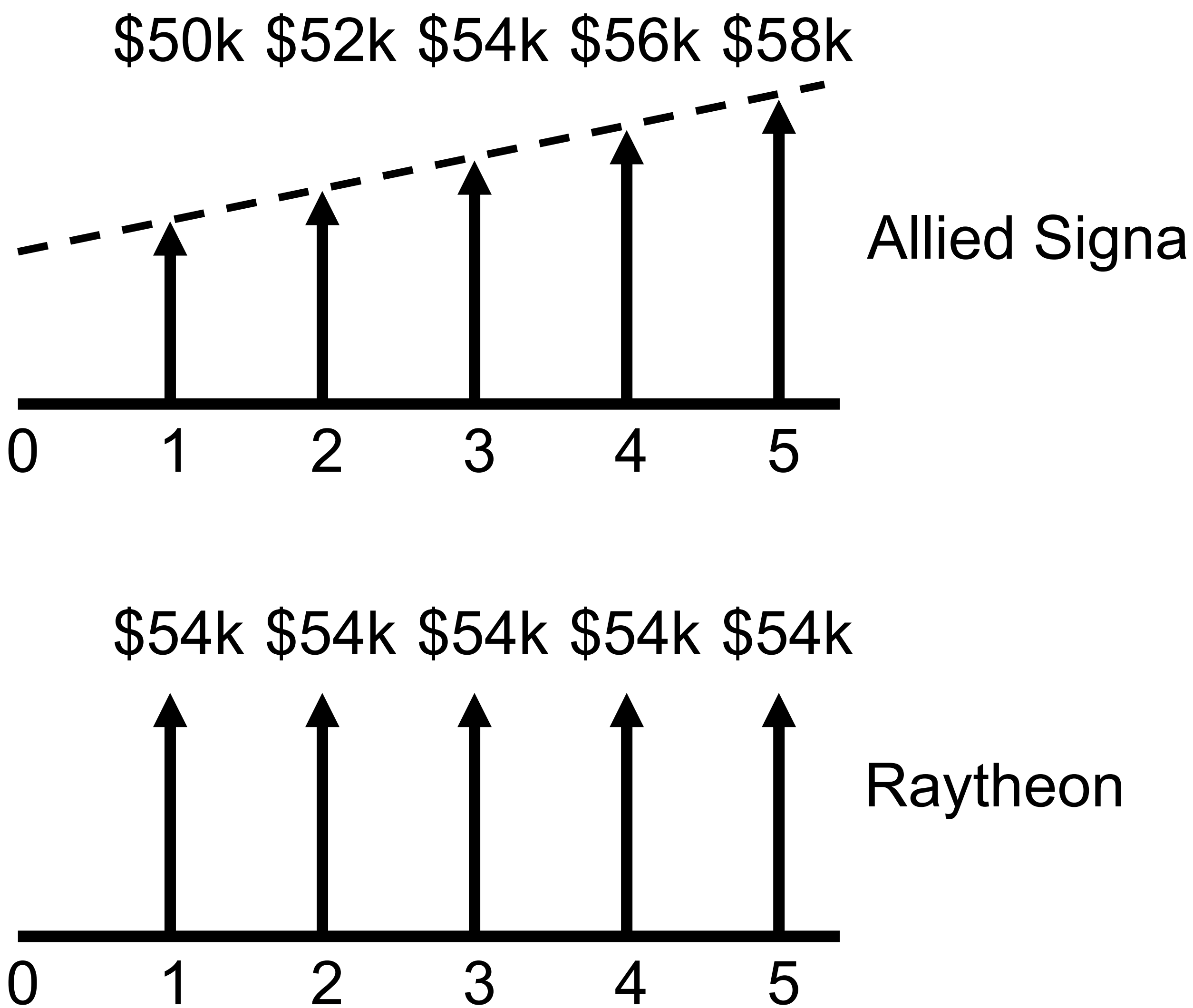
$$P_2 = \$250(P/G, 12\%, 5)$$
$$= \$1{,}599.20$$

$$P = \$3{,}604.08 + \$1{,}599.20$$
$$= \$5{,}204$$

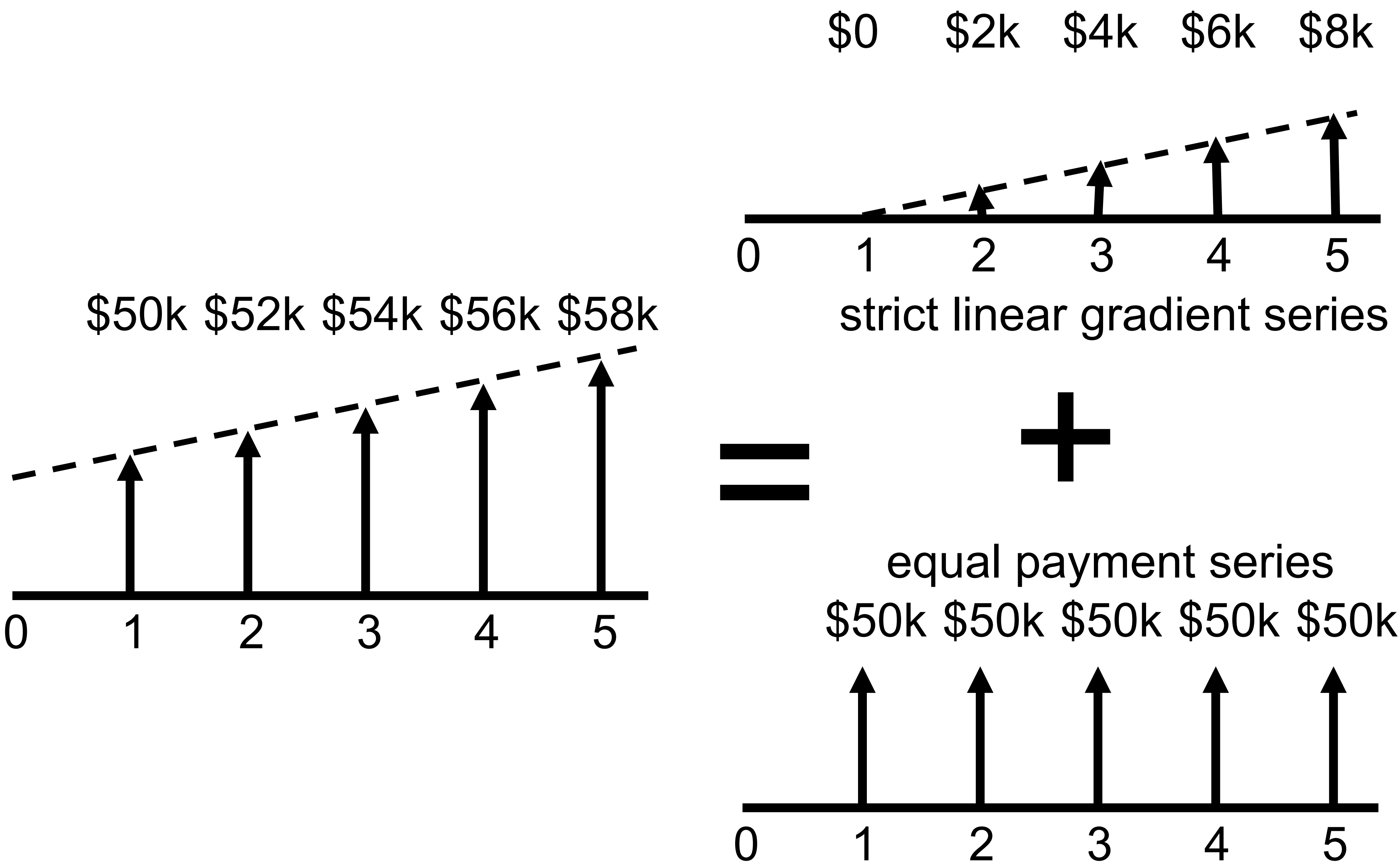
Example – linear gradient

You are trying to decide between 2 job offers. Allied Signal has offered to pay you $50,000/year, with guaranteed pay increases of $2,000/year. Raytheon has offered to start you at $54,000/year, with no pay increases over the next 5 years. What is the *present worth* of the each cash flow over the next 5 years, using the end of year convention and assuming an 8% interest rate is available?

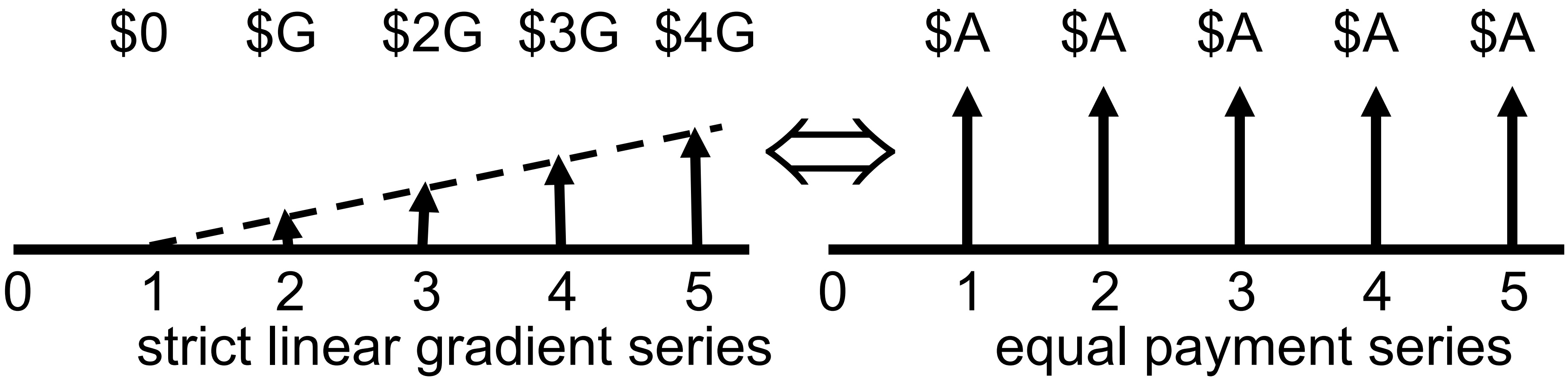
Linear gradient series



Composite cash flow



Gradient to equal-payment conversion



P = G(P/G, i, N)

A = P(A/P, i, N)

A = G(P/G, i, N)(A/P, i, N)

$$A = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right]\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right] = G\left[\frac{(1+i)^N - iN - 1}{i\left[(1+i)^N - 1\right]}\right] = G(A/G, i, N)$$

Future worth of a gradient series

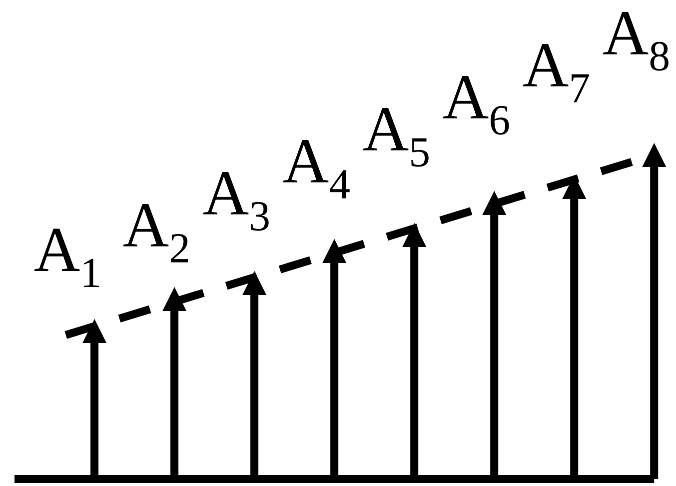
A = G(A/G,i,N)
F = A(F/A,i,N)
F = G(A/G,i,N)(F/A,i,N) = G(F/G,i,N)

$$F = G\left[\frac{(1+i)^N - iN - 1}{i\left[(1+i)^N - 1\right]}\right]\left[\frac{(1+i)^N - 1}{i}\right] = \frac{G}{i}\left[\frac{(1+i)^N - 1}{i} - N\right] = G(F/G,i,N)$$

Linear vs. geometric gradient

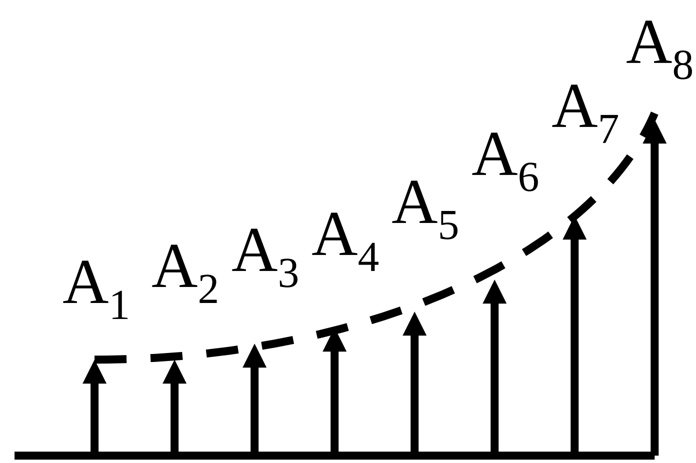
Cash flows on a *linear* gradient increase by a *constant* amount each interest period.

$$A_n = A_1 + (n-1)G, n = 1, 2, ..., n$$



Cash flows on a *geometric* gradient increase by a constant *percentage* each interest period. The percentage is call the *growth rate*, *g*.

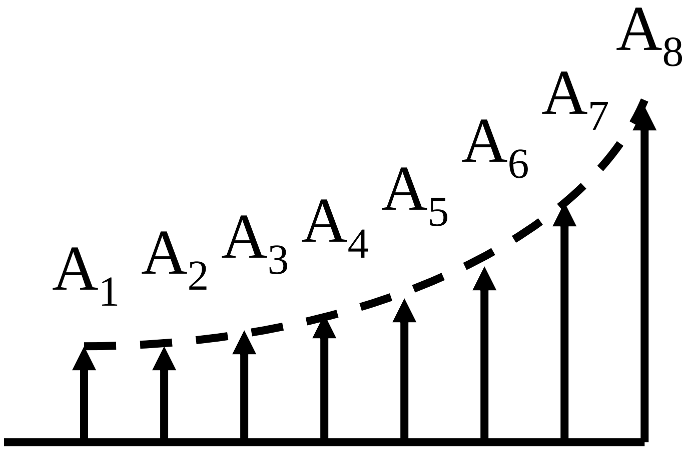
$$A_n = A_1(1+g)^{n-1}, n = 1, 2, ..., n$$



Geometric gradient application

Suppose you bought \$P worth of an income stock, which pays a steady dividend of g% each year, and you reinvest the dividends (to buy more stock). Draw the dividend income cash flow for 8 years. (\$Pg = \$$A_1$)

$$A_n = A_1(1+g)^{n-1}, n = 1, 2, \ldots, n$$



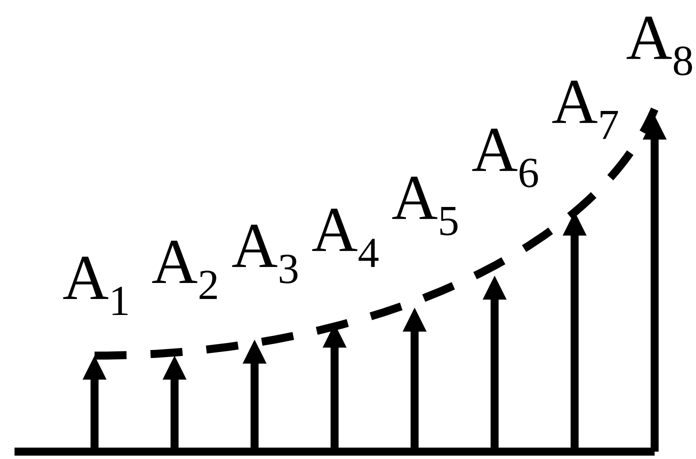
Present worth factor

$$P = A_1 \frac{1 - (1 - g)^N (1 + i)^{-N}}{i - g}$$

$i \neq g$

$$P = A_1 \frac{N}{1 + i}$$

$i = g$



Example – problem 2.43

What is the amount of 10 equal annual deposits that can provide five annual withdrawals, when a first withdrawal of $1000 is made at the end of year 11, and subsequent withdrawals increase at the rate of 6% per year over the previous year's, if…

(a) … the interest rate is 8%, compounded annually.

(b) … the interest rate is 6%, compounded annually.

Example 2.17: find P, given A_1, g, i, N

- given:
  - $g = 5\%$
  - $i = 7\%$
  - $N = 25$ years
  - $A_1 = \$50,000$
- find: P

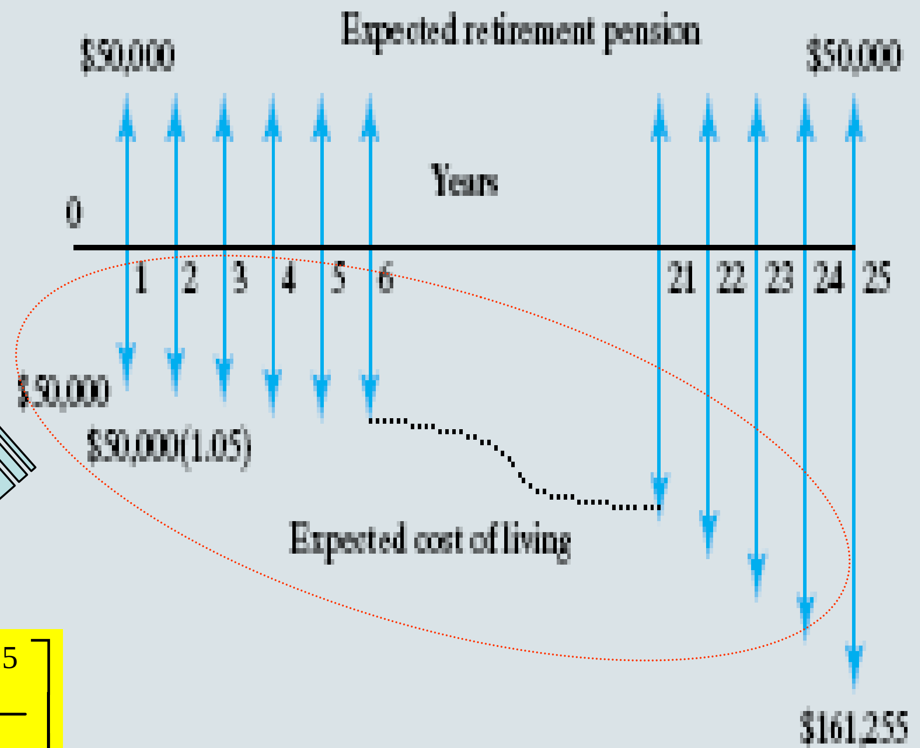


FIGURE 2.30 Cash flow diagram

$$P = \$50,000\left[\frac{1-(1+0.05)^{25}(1+0.07)^{-25}}{0.07-0.05}\right]$$

$$= \$940,696$$

Required additional savings

$$P = \$50,000(P/A, 7\%, 25)$$

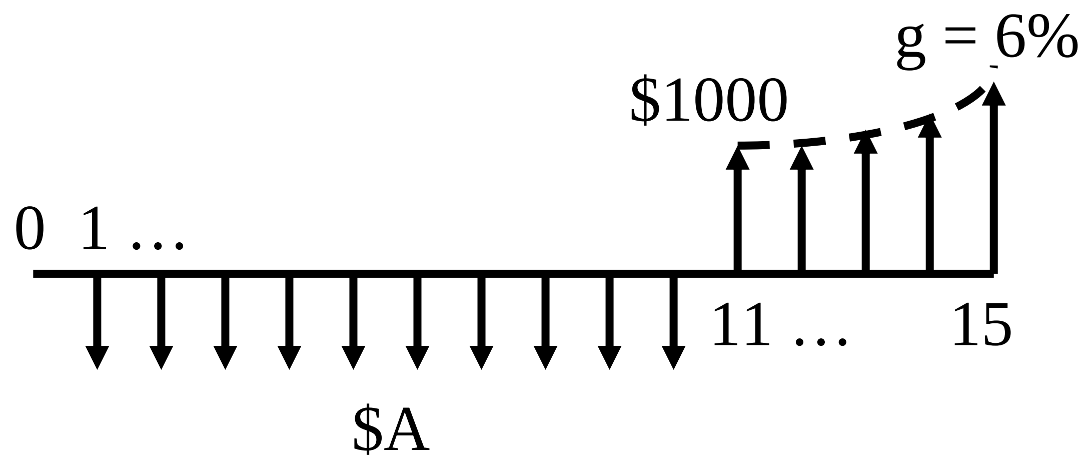
$$= \$582,679$$

$$\Delta P = \$940,696 - \$582,679$$

$$= \$358,017$$

Problem 2.32 Cash Flow

- Equal payment series, A = $?, i = (a) 8% or (b) 6%, N = 10.
- Geometric gradient series, A_1 = $1000, paid at end of interest period 11, g = 6%, N = 5, i = (a) 8% or (b) 6%



Composite cash flows

$$P_{\text{Group 1}} = \$50(P/F, 15\%, 1) = \$43.48$$

$$P_{\text{Group 2}} = \$100(P/A, 15\%, 3)(P/F, 15\%, 1) = \$198.54$$

$$P_{\text{Group 3}} = \$150(P/A, 15\%, 4)(P/F, 15\%, 4) = \$244.85$$

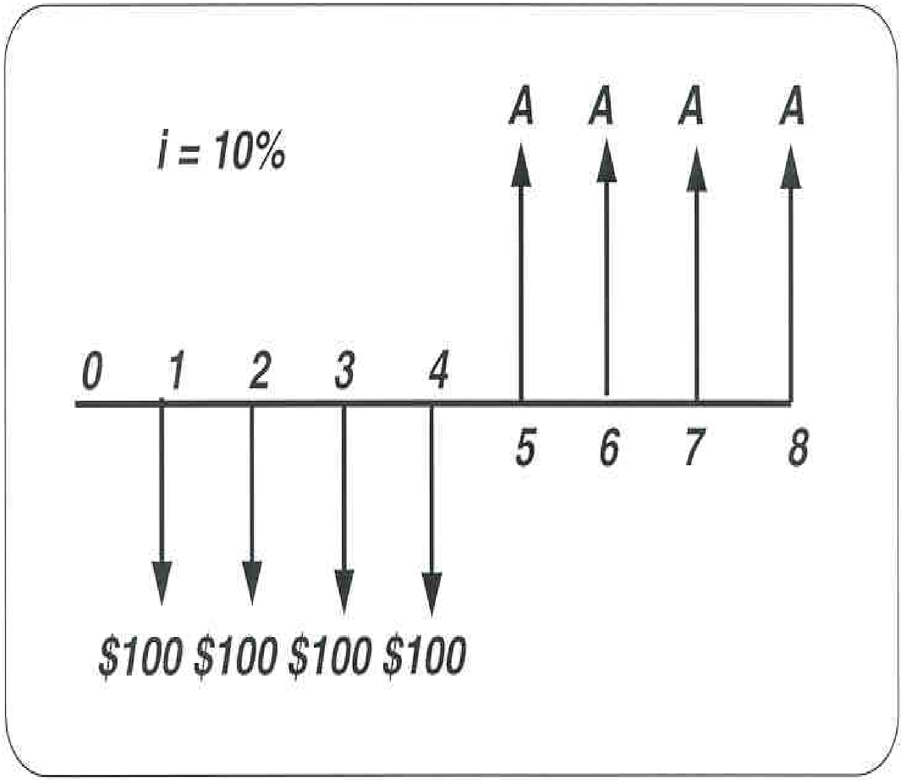
$$P_{\text{Group 4}} = \$200(P/F, 15\%, 9) = \$56.85$$

$$P = \$43.48 + \$198.54 + \$244.85 + \$56.85 = \$543.72$$

Figure 2-32

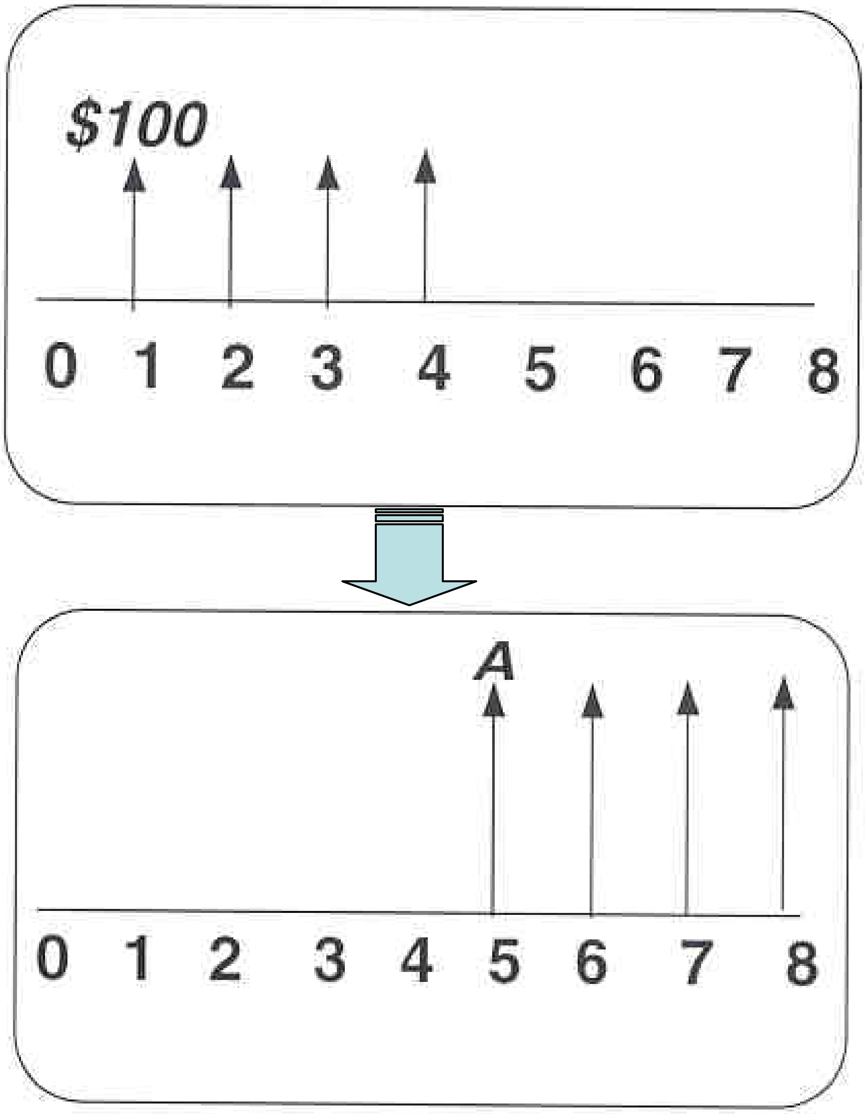
Unconventional equivalence calculations

Situation 1: If you make 4 annual deposits of $100 in your savings account which earns 10% annual interest, what equal annual amount can be withdrawn over 4 subsequent years?

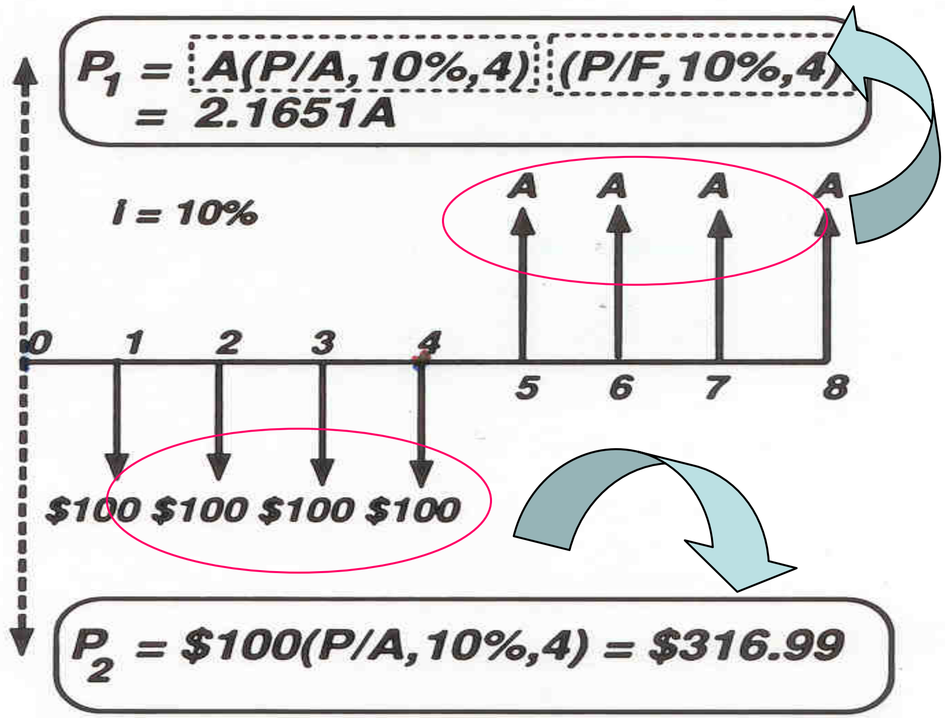


Unconventional equivalence calculations

Situation 2: What value of *A* would make the two cash flow transactions equivalent if $i = 10\%$?



Method 1: Establish economic equivalence at period 0

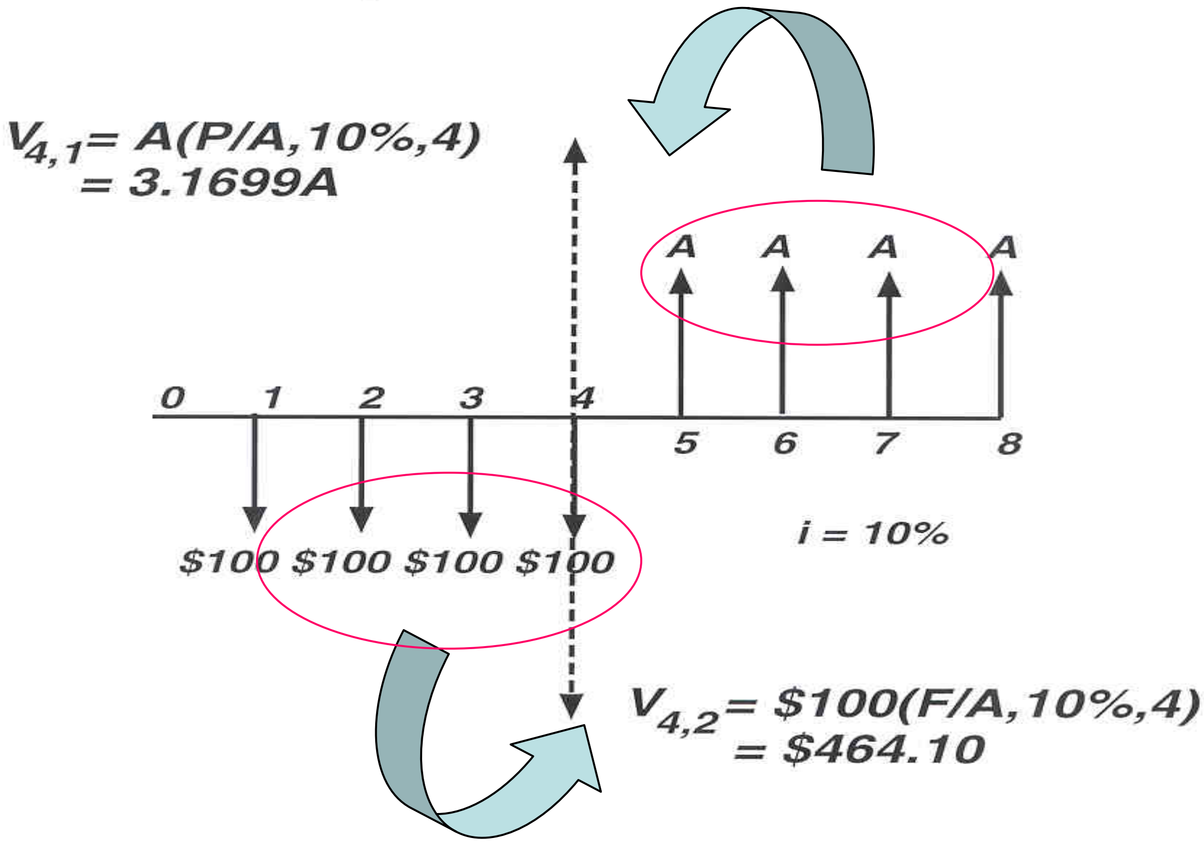


$$P_1 = P_2$$

$$2.1651A = \$316.99$$

$$A = \$146.41$$

Method 2: Establish economic equivalence at period 4



$3.1699A = \$464.10$

$A = \$146.41$

Example – problem 2.53

Find the value of *X* so that the following 2 cash flows are equivalent for an interest rate of 10% compounded annually: (i) $400 withdrawn in each of years 1-3 and $200 withdrawn in each of years 4-5, and (ii) $*X* withdrawn in each of years 1-2, 5 and $300 deposited in each of years 3-4. See 2.53 in text for figure.